From the unitary condition on S, we have

$$(S^*S)_{i,i+1} = \sum_{k=1}^{n} s_{k,i}s_{k,i+1} = 0$$

and therefore $s_{i-1,i}^* \; s_{i-1,i+1} + s_{i,i}^* \; s_{i,i+1} = 0$

Substituting in (21) gives

$$x_{i-1,i+1} = 0 \quad \text{for all } i.$$

Hence the matching at port i isolates port $i+1$ from port $i-1$.

III. Conclusions

It has been shown that a certain class of three-port network can be transformed into a circulator by the addition of an appropriate reactive discontinuity at each port. This transformation is possible if the three-port junction is loss-free and such that (for suitable port numbering) the transmission from port 1 to 2 is greater than from 2 to 1, from port 2 to 3 is greater than 3 to 2, and from 3 to 1 is greater than from 1 to 3. When the three-port network is symmetrical, the restriction reduces to the network, being loss-free and the moduli of the two transmission coefficients being different.

The important consequence of the proposed synthesis is that it is not necessary to obtain complete matching by means of the ferrite junction configuration and applied magnetic field, but that external reciprocal elements may be used, which have, of course, predictable characteristics. Hence, an approach to the design of broad-band circulators is proffered.

It has also been shown that by the use of small reactive discontinuities, an imperfect (but loss-free) n-port circulator can be matched to make vanish, simultaneously, the reflection and one transmission coefficient at each port. The effect of these small discontinuities on the other transmission coefficients is of second order of smallness, so that they cannot generally be made zero. It follows that the isolation of a practical four-port circulator must be optimized within the junction, but any small remaining reflection and cross-coupling can be matched externally by reactive discontinuities.

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Propagation of Surface Waves on an Inhomogeneous Plane Layer*

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Summary—**The permittivity of a plane layer is assumed to vary continuously as a function of distance measured from the surface. Solutions for the field distributions of surface waves on the inhomogeneous layer are developed with the WKB technique. Transcendental equations for the phase velocity are derived for TE and TM modes. These equations are solved most conveniently with the aid of phase-velocity graphs which are included. The accuracy of the solution is verified by comparison with the rigorous solution for an exponential inhomogeneity.**

Introduction

The permittivity of most radome materials changes by a significant amount when the temperature is increased by hypersonic flight through the atmosphere. The outer surface of the radome becomes hotter than the inner, resulting in a continuous variation in permittivity even if the radome was designed as a homogeneous structure.

Moreover, new techniques of radome fabrication may make feasible the construction of continuously inhomogeneous radomes. This can be accomplished with variable loading or with variable density foams. Alternatively, a multilayer sandwich having many thin laminations can form an adequate approximation. These structures may have a greater bandwidth or may allow a greater range of incidence angles than conventional radomes.

The characteristics of surface waves on inhomogeneous layers are of interest to the radome designer because he must minimize the excitation of these waves and their deletereous effects on the radar system performance. The antenna designer is interested in the effects of unintentional inhomogeneities, arising from thermal gradients, on the performance of surface-wave antennas, and the advantages that may accrue from the use of intentional inhomogeneities in such antennas.

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Exact solutions in closed form are available only for a few special cases including the linear and exponential inhomogeneities. Step-by-step numerical integration can be applied to the analysis of surface waves on inhomogeneous layers only on a trial-and-error basis.

A more convenient and useful solution, the WKB solution (Wentzel, Kramers, and Brillouin [1]), is developed for the field distribution of surface waves on inhomogeneous plane layers. These expressions are employed in deriving the transcendental equations for the phase velocity. These can be solved most conveniently by means of the graphs which are included. The accuracy of these results is demonstrated by comparison with the exact solutions for exponentially inhomogeneous layers. Finally, the limitations of the WKB solution are discussed for layers having rapid variations in permittivity.

Theory

General

Defining a coordinate system as in Fig. 1, consider the waves which can exist on a lossless, isotropic, inhomogeneous layer and have the following properties:

1) Harmonic time dependence $e^{j\omega t}$,
2) Plane phase fronts (space dependence e^{-jhz}),
3) No dependence on the y-coordinate, and
4) Exponential decay $e^{-\alpha x}$ normal to the layer.

The dielectric layer is assumed to be infinitely wide but of finite thickness d. Media I and III are homogeneous with permeability μ_0 and permittivity ϵ_0. The permeability μ of region II is assumed constant, but it may differ from μ_0. The permittivity $\epsilon(x)$ may vary continuously as a function of distance from the center of the layer. It is assumed to be an even function of x and discontinuous at $x=\pm a$.

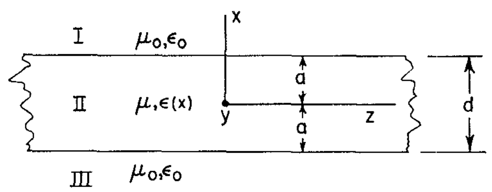

Fig. 1—An inhomogeneous plane layer and the coordinate system.

TE Surface Waves

For the TE surface waves it is assumed that the electric field intensity vector $\boldsymbol{E}$ is parallel with the y-axis and is given by

$$E_y = \begin{cases} e^{-\alpha x}e^{-jhz} & \text{in region I} \\ f(x)e^{-jhz} & \text{in region II} \\ e^{\alpha x}e^{-jhz} & \text{in region III} \end{cases} \tag{1}$$

and

$$E_x = E_z = 0. \tag{2}$$

From Maxwell's differential equations it follows that

$$H_y = 0 \tag{3}$$

and

$$H_z = \frac{j}{\omega\mu}\frac{\partial E_y}{\partial x} = \begin{cases} (-j\alpha/\omega\mu_0)e^{-\alpha x}e^{-jhz} & \text{in region I} \\ (j/\omega\mu)f'(x)e^{-jhz} & \text{in region II} \\ (j\alpha/\omega\mu_0)e^{\alpha x}e^{-jhz} & \text{in region III} \end{cases} \tag{4}$$

where

$$f'(x) = df/dx. \tag{5}$$

The boundary conditions require that E_y and H_z be continuous at $x=0$, $x=a$ and $x=-a$. These conditions lead to the following:

$$f(a) = e^{-\alpha a} \tag{6}$$

$$f'(a) = -\alpha\mu_r e^{-\alpha a} \tag{7}$$

$$f'(0) = 0 \tag{8}$$

where

$$\mu_r = \mu/\mu_0. \tag{9}$$

Furthermore, Maxwell's equations yield the following:

$$f''(x) + (\omega^2\mu\epsilon - h^2)f(x) = 0. \tag{10}$$

If the permittivity is a slowly varying[1] function of x, the WKB [1] solution for (10) is useful. It can be written in the form

$$f(x) = \frac{A\cos\int_x^a \sqrt{\omega^2\mu\epsilon - h^2}\,dx + B\sin\int_x^a \sqrt{\omega^2\mu\epsilon - h^2}\,dx}{(\omega^2\mu\epsilon - h^2)^{1/4}}. \tag{11}$$

The constant coefficients A and B are determined by use of (6) and (7). In performing the differentiation indicated in (7), it is assumed that the numerator in (11) is a much more rapidly varying function of x than the denominator. This leads to the following:

$$A = (\omega^2\mu\epsilon_a\epsilon_0 - h^2)^{1/4}e^{-\alpha a} \tag{12}$$

$$B = \mu_r\alpha e^{-\alpha a}/(\omega^2\mu\epsilon_a\epsilon_0 - h^2)^{1/4} \tag{13}$$

where ϵ_a is the relative permittivity at the edge of the inhomogeneous layer (at $x=a$).

Eqs. (8), (11), (12) and (13) yield the following transcendental equation for the TE surface waves:

$$\tan k\int_0^a \sqrt{\mu_r\epsilon_r - h^2/k^2}\,dx = \frac{\mu_r\alpha/k}{\sqrt{\mu_r\epsilon_a - h^2/k^2}} \tag{14}$$

where

$$k = \omega\sqrt{\mu_0\epsilon_0}. \tag{15}$$

[1] Eq. (44) gives a more precise statement of the limitations of the WKB solution for layers having rapid variations in permittivity.

When the wave equation is applied to the field expressions in region I it is found that

$$\alpha^2 - h^2 = -k^2. \tag{16}$$

It is possible to solve (14) and (16) to determine the surface-wave phase constant h, or h/k. A convenient method of accomplishing this is developed in a later section.

TM Surface Waves

Since the derivation for the TM surface waves is similar to that for the TE waves, it will suffice to list the equations below.

$$H_x = H_z = E_y = 0. \tag{17}$$

$$H_y = \begin{cases} e^{-\alpha x}e^{-jhz} & \text{in region I} \\ f(x)e^{-jhz} & \text{in region II} \\ e^{\alpha x}e^{-jhz} & \text{in region III} \end{cases} \tag{18}$$

$$E_z = \begin{cases} (j\alpha/\omega\mu_0)e^{-\alpha x}e^{-jhz} & \text{in region I} \\ (-j/\omega\epsilon)f'(x)e^{-jhz} & \text{in region II} \\ (-j\alpha/\omega\epsilon_0)e^{\alpha x}e^{-jhz} & \text{in region III} \end{cases} \tag{19}$$

$$f(a) = e^{-\alpha a} \tag{20}$$

$$f'(a) = -\alpha\epsilon_a e^{-\alpha a} \tag{21}$$

$$f'(0) = 0 \tag{22}$$

$$f(x) = \sqrt{\epsilon_r}\,\frac{A\cos\int_x^a \sqrt{\omega^2\mu\epsilon - h^2}\,dx + B\sin\int_x^a \sqrt{\omega^2\mu\epsilon - h^2}\,dx}{(\omega^2\mu\epsilon - h^2)^{1/4}} \tag{23}$$

$$A = (\omega^2\mu\epsilon_a\epsilon_0 - h^2)^{1/4}e^{-\alpha a}/\sqrt{\epsilon_a} \tag{24}$$

$$B = \alpha\sqrt{\epsilon_a}\,e^{-\alpha a}/(\omega^2\mu\epsilon_a\epsilon_0 - h^2)^{1/4} \tag{25}$$

$$\tan k\int_0^a \sqrt{\mu_r\epsilon_r - h^2/k^2}\,dx = \frac{\epsilon_a\alpha/k}{\sqrt{\mu_r\epsilon_a - h^2/k^2}} \tag{26}$$

$$\alpha^2 - h^2 = -k^2. \tag{27}$$

Since $E_z = 0$ at $x = 0$, a metal plane may be inserted at $x = 0$ without disturbing the fields of the TM mode. Thus, the equations apply equally well for a layer of thickness a on a metal plane and a symmetrical layer of total thickness $2a$ without a metal plane.

Exact Solution for Exponential Inhomogeneity, TE Case

If the permittivity varies with x in accordance with

$$\mu_r\epsilon_r(x) = h^2/k^2 + ce^{b|x|}, \tag{28}$$

it will be referred to as an "exponential inhomogeneity," and (10) reduces to

$$f''(x) + k^2ce^{b|x|}f(x) = 0. \tag{29}$$

The quantities μ_r, h, k, c and b are independent of x. The exact solution of (29) is given by

$$f(x) = FJ_0(\psi) + GY_0(\psi) \tag{30}$$

where

$$\psi = 2k\sqrt{c}\,e^{bx/2}/|b|. \tag{31}$$

The coefficients F and G can be determined by means of (6) and (7), yielding

$$F = [\alpha\mu_r \operatorname{sgn}(b)Y_0^a - k\sqrt{c}\,e^{ba/2}Y_1^a]\pi e^{-\alpha a}/|b| \tag{32}$$

$$G = [-\alpha\mu_r \operatorname{sgn}(b)J_0^a + k\sqrt{c}\,e^{ba/2}J_1^a]\pi e^{-\alpha a}/|b| \tag{33}$$

where superscript a indicates an argument ψ evaluated at $x = a$, and sgn (b) is unity if b is positive and -1 if b is negative. Eqs. (8) and (30) yield

$$\frac{\alpha\mu_r \operatorname{sgn}(b)}{k\sqrt{c}\,e^{ba/2}} = \frac{J_1^oY_1^a - J_1^aY_1^o}{J_1^oY_0^a - J_0^aY_1^o} \tag{34}$$

where the superscript o indicates an argument ψ evaluated at $x = 0$. Furthermore,

$$\alpha^2 - h^2 = -k^2. \tag{35}$$

Eqs. (34) and (35) can be employed to calculate the phase constant h for TE surface waves on the inhomogeneous layer described by (28).

It should be pointed out that this rigorous solution applies to a plane layer whose permittivity is the sum of an exponential function and a constant, *provided that* this constant happens to be the square of the relative phase constant [see (28)].

Graphical Solution

It is interesting to compare the transcendental equations for surface waves on homogeneous and inhomogeneous layers. For TE surface waves on a homogeneous layer of thickness $2a'$, relative permeability μ_r and relative permittivity ϵ_a [2],

$$\mu_r\sqrt{\frac{h^2/k^2 - 1}{\mu_r\epsilon_a - h^2/k^2}} = \tan ka'\sqrt{\mu_r\epsilon_a - h^2/k^2}. \tag{36}$$

For TE surface waves on an inhomogeneous layer of thickness $2a$, relative permeability μ_r and relative permittivity $\epsilon_r(x)$, (14) and (16) yield

$$\mu_r\sqrt{\frac{h^2/k^2 - 1}{\mu_r\epsilon_a - h^2/k^2}} = \tan k\int_0^a \sqrt{\mu_r\epsilon_r - h^2/k^2}\,dx. \tag{37}$$

A comparison of these two equations shows that the phase constant h is the same (in the WKB approximation) for the homogeneous and inhomogeneous layers if

$$a' = \frac{\int_0^a \sqrt{\mu_r\epsilon_r - h^2/k^2}\,dx}{\sqrt{\mu_r\epsilon_a - h^2/k^2}}. \tag{38}$$

Obviously (37) is satisfied by that value of h/k which satisfies both (36) and (38). The solution of (36) is plotted in Fig. 2 as a curve of h/k vs a'/λ_0 for a TE wave on a homogeneous layer having $\mu_r\epsilon_a = 8$. (λ_0 represents the wavelength in free space.) Given the permittivity function $\epsilon_r(x)$ for an inhomogeneous layer, it is possible to plot the solution of (38) on the same graph by choosing several numerical values for h/k and carrying out the indicated integration to determine the corresponding values of a'. The phase constant h (or h/k) for the inhomogeneous layer is then determined from the intersection of the two curves.

The transcendental equation for TM surface waves on a homogeneous layer is [2]

$$\epsilon_a \sqrt{\frac{h^2/k^2 - 1}{\mu_r\epsilon_a - h^2/k^2}} = \tan ka' \sqrt{\mu_r\epsilon_a - h^2/k^2}. \qquad (39)$$

From (26) and (27), the transcendental equation for TM surface waves on an inhomogeneous layer is

$$\epsilon_a \sqrt{\frac{h^2/k^2 - 1}{\mu_r\epsilon_a - h^2/k^2}} = \tan k \int_0^a \sqrt{\mu_r\epsilon_r - h^2/k^2}\, dx. \qquad (40)$$

A comparison of (39) and (40) shows that (40) is satisfied by that value of h/k which satisfies both (38) and (39). The solution of (39) is plotted in Fig. 2 as a curve of h/k vs a'/λ_0 for TM waves on a homogeneous layer having $\mu_r\epsilon_a = 8$. The phase constant for an inhomogeneous layer is determined by the intersection of this curve with the graph of (38).

To illustrate the results that may be expected from the WKB formulas, three examples are listed in Table I.

The locus of solutions of (38) is plotted in Fig. 2 for Example 1. The trapezoidal rule, Simpson's rule, or tables of integrals may be used to evaluate a' in (38) for several values of h/k. The resulting data points (h/k, a') are entered on Fig. 2 and connected by the curve labeled "Eq. 38." The intersection with the TE_0 curve yields $h/k = 2.565$ as listed in Table I, and the intersection with the TM_0 curve yields $h/k = 2.490$. The exact solutions listed in Table I were obtained by letting $\psi(o) = 1$ and $\psi(a) = 2.3522$ for Example 1, and $\psi(o) = 5$ and $\psi(a) = 6.2964$ for Example 2. $\psi(x)$ is defined by (31).

For TE waves, comparison of (36) and (37) shows that the WKB solution reduces to the exact solution in the special case of the homogeneous layer. For TM waves (39) and (40) show that the WKB solution is exact for the homogeneous layer. In the derivation of the WKB formulas it is assumed that the permittivity is a slowly varying function. In particular, it is assumed that

$$\mu_r \left| \epsilon_r' \right| \ll 2k(\mu_r\epsilon_r - h^2/k^2)^{3/2} \qquad (41)$$

everywhere except at the edges where the permittivity is discontinuous. (The prime indicates differentiation with respect to x.) Accurate results may be anticipated only if this inequality is satisfied.

In Examples 1, 2 and 3 (Table I), inequality (44) is satisfied by a factor of 1.00, 5.00 and infinity, respectively. It may be noted in Table I that accurate results are obtained if (44) is satisfied.

Fig. 3 shows the exact and WKB solutions for the

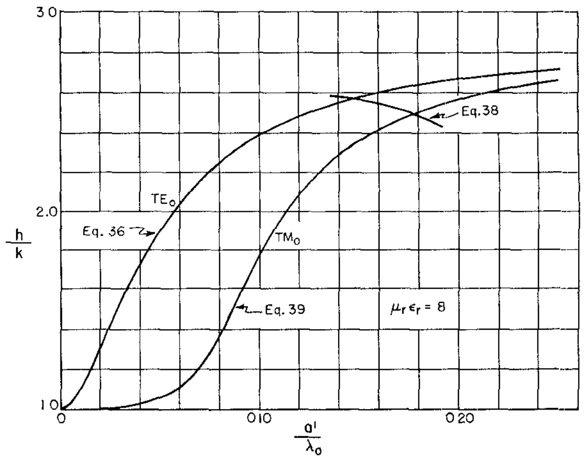

Fig. 2—Relative phase constant vs thickness for TE_0 and TM_0 surface waves on a homogeneous dielectric layer of thickness $2a'$.

TABLE I

Phase Velocity of TE_0 Mode on Inhomogeneous Layer Having $a = 0.25\lambda_0$

Example	$\mu_r\epsilon_r(x)$	$\mu_r\epsilon_r(0)$	$\mu_r\epsilon_r(a)$	h/k WKB	h/k Exact
1	$0.2966e^{1.0892k\lvert x\rvert}+6.3590$	6.6556	8	2.565	2.5217
2	$0.5388e^{0.2936k\lvert x\rvert}+7.1455$	7.6843	8	2.685	2.6731
3	8.0	8.0	8	2.71	2.71

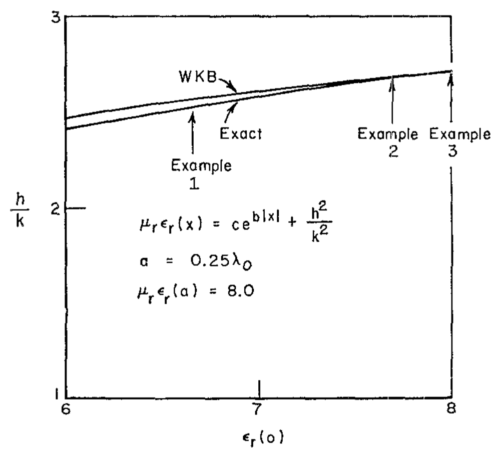

Fig. 3—WKB and exact solutions for the relative phase velocity on an exponentially inhomogeneous layer.

relative phase velocity on exponentially inhomogeneous layers such as those listed in Table I.

Additional curves of phase constant vs thickness are available for homogeneous layers having $\mu_r\epsilon_r = 2$, 4, 8 and 15 [3]. These may be employed in solving for the phase constants of inhomogeneous layers having $\mu_r\epsilon_a = 2$, 4, 8 and 15 with the technique described herein.

Conclusion

The permittivity of a plane layer is assumed to vary continuously as a function of distance measured from the surface. Solutions for the field distributions of surface waves on the inhomogeneous layer are developed with the WKB technique. Transcendental equations for the phase velocity are derived for TE and TM modes. These equations are solved most conveniently with the aid of phase-velocity graphs which are included. The accuracy of the solution is verified by comparison with the rigorous solution for an exponential inhomogeneity.

Reasonably good accuracy is obtained even when the relative permittivity varies from 6 to 8 in a distance of 0.25 wavelength.

The formulas presented herein reduce to the rigorous solution for homogeneous layers and are accurate if the permittivity gradient is small at each point within the layer.

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UHF Backward-Wave Parametric Amplifier*

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Summary—**This paper describes a breadboard model of a UHF varactor diode backward-wave parametric amplifier that can be electronically tuned over an octave tuning range (250–500 Mc). It operates in a mode that has a relatively constant idler frequency; however, it uses two forward-wave transmission lines in contrast to the backward-wave transmission line requirement previously reported.**

A theoretical discussion on the design considerations of this mode is presented and applied to the UHF model. Measurements taken in the conventional mode of operation (output frequency equal to the input frequency) yielded voltage gain bandwidth products in excess of 100 Mc and over-all effective receiver noise temperatures of less than 140°K. Detailed measurements in the mode where the constant idler frequency is used as the output were not taken because directional filters and circulators, which are necessary in this mode, were not available.

I. Introduction

THE BACKWARD-WAVE parametric amplifier (BWPA) is a low-noise preamplifier that is capable of being electronically tuned at a rapid rate over a greater-than-octave tuning range [1]–[3]. It consists, in general, of two separate and distinct circuits that are coupled together by means of nonlinear or time-varying reactive elements.

Recently, a new class of BWPA has been evolved [4], [5] in which the center frequency of the output pass band (which is taken at the idler frequency) remains constant as the input amplification band of the amplifier is varied. This is an advantage over the conventional BWPA since it eliminates the tracking problems associated with the complex demodulator necessary to convert the normally varying output frequency to a constant IF. It thus yields an amplification system that has a greater tuning rate potential than that of the conventional BWPA. However, the realization of this amplifier mode required one of the two coupled transmission lines to have a backward-wave characteristic, which at the lower frequencies does not present any problems but presents increasing design difficulty as the frequency approaches the UHF and microwave region.

This paper proposes a new configuration that yields a nearly constant idler frequency over an octave tuning range in which both of the coupled transmission lines are forward-wave types and it presents theoretical and experimental results of a UHF model whose design was based upon this configuration.

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